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Further Reflections on Thompson's Conjecture*

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Let G be a finite group and let $N(G) = \{n \in N \mid G$ has a conjugacy class C, such that $|C| = n\}$. Professor J. G. Thompson has conjectured that: If G is a finite group with $Z(G) = 1$ and M a non-abelian simple group satisfying $N(G) = N(M)$, then $G \cong M$.

We have proved previously that: If M is a sporadic simple group or a simple group having its prime graph with at least three prime graph components, then Thompson's conjecture is correct. In this paper, we shall prove:

MAIN THEOREM. Let G be a finite group with $Z(G) = 1$ and $M = G_2(q)$ or $G_2(2)'$, where $q \geq 2$, such that $N(G) = N(M)$. Then $G \cong M$.

Key Words: Finite groups; conjugacy classes; characterization of a finite simple group

All notation are the same as in [4].

The next two lemmas follow from Lemmas 1.4 and 1.5 in [4].

LEMMA 1. *Suppose G and M are two finite groups satisfying $t(M) \geq 2$, $N(G) = N(M)$, and $Z(G) = 1$. Then $|G| = |M|$.*

LEMMA 2. *Let G_1 and G_2 be finite groups satisfying $|G_1| = |G_2|$ and $N(G_1) = N(G_2)$. Then $t(G_1) = t(G_2)$ and $T(G_1) = T(G_2)$.*

The next lemma follows from Theorem 2 in [2].

LEMMA 3. *Suppose G is a Frobenius group of even order and H and K are the Frobenius kernel and the Frobenius complement of G, respectively. Then*

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$t(G) = 2$, $T(G) = \{\pi(H), \pi(K)\}$, and G has one of the following structures:

(1) $2 \in \pi(H)$, *all Sylow subgroups of K are cyclic.*

(2) $2 \in \pi(K)$, *H is an abelian group, K is a solvable group, the Sylow subgroups of odd order of G are cyclic groups, and the 2-Sylow subgroups of G are cyclic or generalized quaternion groups.*

(3) $2 \in \pi(K)$, *H is an abelian group, and there exists $K_0 \le K$ such that $[K : K_0] \le 2$, $K_0 = \mathbf{Z} \times SL(2,5)$, $(|Z|, 2 \times 3 \times 5) = 1$,*

and the Sylow subgroups of $\mathbf{Z}$ are cyclic.

The next lemma follows from Theorem 2 in [2].

LEMMA 4. *Let G be a 2-Frobenius group of even order. Then $t(G) \ge 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(K/H) = \pi_2$, $\pi(H) \cup \pi(G/K) = \pi_1$, $|G/K| \mid |\mathrm{Aut}(K/H)|$, and G/K and K/H are cyclic. Especially, $|G/K| < |K/H|$, and G is solvable.*

LEMMA 5. *If G is a finite group with more than one prime graph component, then G is one of the following*: (a) *Frobenius or 2-Frobenius,* (b) *simple,* (c) *an extension of a π_1-group by a simple group,* (d) *simple by π_1-solvable, or* (e) *a π_1-group by simple by π_1-group, where π_1 is the prime graph component containing prime 2* (*see* [8]).

LEMMA 6. *If G is a finite group with more than one prime graph component and has a normal series $H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is simple, then H is a nilpotent group* (*see Lemma 3 in* [8]).

LEMMA 7. *Let G be a finite group with $Z(G) = 1$. If M is a non-abelian simple group with $t(M) = 2$ satisfying $N(G) = N(M)$, then*

(1) $|G| = |M|$, *$t(G) = t(M)$, and $T(G) = T(M)$.*

(2) *Let $|M| = m_1 m_2$, $T(M) = \{\pi_1, \pi_2\}$, and $\pi(m_i) = \pi_i$ for $i = 1$ or 2. Then $|G| = m_1 m_2$ and one of the following holds*:

(a) *G is a Frobenius or 2-Frobenius group*;

(b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group, and K/H is a non-abelian simple group. Moreover $T(K/H) = \{\pi'_1, \pi'_2, \ldots, \pi'_s, \pi_2\}$, $|K/H| = m'_1 m'_2 \cdots m'_s m_2$ and $m'_1 m'_2 \cdots m'_s | m_1$ where $\pi(m'_j) = \pi'_j$, $1 \le j \le s$.*

(3) $|G/K| \,\big|\, |\mathrm{Out}(K/H)|$.

Proof. (1) and (2) follow from Lemmas 1–6. Since $t(G) \ge 2$, we have $t(G/H) \ge 2$. Otherwise $t(G/H) = 1$, so that $t(G) = 1$ since $2 \mid |H|$ and H

a π_1-group, a contradiction. Moreover we have $Z(G/H) = 1$. For any $xH \in G/H$ and $xH \notin K/H$, xH induces an automorphism of K/H. And this automorphism is trivial iff $xH \in Z(G/H)$. Therefore $G/K \leq \mathrm{Out}(K/H)$ for $Z(G/H) = 1$, from which (3) follows.

Let G be a finite group of even order and $T(G) = \{\pi_1, \pi_2, \dots, \pi_t\}$, while $2 \in \pi_1$. Then $|G| = m_1 m_2 \cdots m_t$ such that $\pi(m_i) = \pi_i$, $i = 1, 2, \dots, t$. These m_i are called order components of G in [3, 4]. We use $\mathrm{OC}(G)$ to denote the set of order components of G. And by [6, 8] we can easily calculate the order components of simple groups M with $t(M) \geq 2$ (see Tables 1–4 in [3]).

LEMMA 8. *Let G be a finite group with $t(\Gamma(G)) \geq 2$ and N a normal subgroup of G. If N is a π_i-group for some prime graph component π_i of G and $a_1, a_2, \dots, a_r$ are some order components of G but not a π_i number, then $a_1 a_2 \cdots a_r$ is a divisor of $|N| - 1$.*

Proof. For any $x \in N$ and $x \neq 1$, the centralizer of x in G must be a π_i-group and so $a_1, a_2, \dots, a_r$ divides $|\mathrm{Cl}_G(x)|$. Therefore $a_1 a_2 \cdots a_r$ divides $|N| - 1$ for $N \trianglelefteq G$ and N is a union of conjugacy classes.

In the following lemma we use $D(G)$ to denote the odd order component of G. For $M = G_2(q)$ we have $D(M) = q^2 + q + 1$ for $3 | q + 1$ and $q^2 - q + 1$ for $3 | q - 1$.

LEMMA 9. *Let $M = G_2(q)$, $q \geq 4$, $q \equiv -1$ or 1 (mod 3). Let $S_p \in \mathrm{Syl}_p(M)$, where $p \in \pi(M)$ and $p \nmid q$. Then*

(1) *$|S_p| < q^3$ or $|S_p| = p^6$.*

(2) *If $p \in \pi_1(M)$ and $p^\alpha \mid |M|$, then $p^\alpha - 1 \equiv 0 \pmod{D(M)}$ if and only if $p | q$ or $q = 4$, $p^\alpha = 27$ or $q = 5$, $p^\alpha = 32$, where $\pi_1(M)$ is the prime graph component containing 2.*

Proof. Since $|G| = q^6(q^2 - q + 1)(q^2 + q + 1)(q - 1)^2(q + 1)^2$, it is easy to show that (1) holds.

To prove (2), it is sufficient to prove the necessity. And in the rest of the proof, we assume that $p \nmid q$. Since $p^\alpha - 1 \equiv 0 \pmod{D(M)}$, $p^\alpha > D(M)$.

Case 1. If $q \equiv 1 \pmod 3$, then p^α is a factor of $4(q - 1)^2$, $4(q + 1)^2$, $3(q - 1)^2$, and $(q^2 + q + 1)/3$.

(a) If $p^\alpha | 4(q - 1)^2$, then, since $3 | q - 1$, we have $p^\alpha \mid [(q - 1)^2$ or $p^\alpha \mid 4(q -)^2/9$. Thus $p^\alpha < q^2 - q + 1 = D(G)$, a contradiction.

(b) Suppose $p^\alpha | 4(q + 1)^2$.

First we further assume that $p = 2$. If $q + 1$ is not a power of 2, then, since $3 \nmid q + 1$, $p^\alpha \leq 4(q + 1)^2/25 < D(G)$, a contradiction. If $q + 1$ is a power of 2, then $p^{\alpha} = (q+1)^2$, $2(q+1)^2$, or $4(q+1)^2$ for $p^{\alpha} > D(G)$ and $(q+1)^2/2 < D(M)$.

If $p^{\alpha} = (q+1)^2$, then, since $(q+1)^2 - 1 = D(G) + 2q - 1$ and $p^{\alpha} - 1 \equiv 0 \pmod{D(G)}$, we have $D(G) = 2q - 1$. Thus $q = 2$, a contradiction.

If $p^{\alpha} = 2(q+1)^2$, then $p^{\alpha} - 1 = 2D(G) + 6q - 1$. It is easy to check that this is impossible by $D(G) \mid p^{\alpha} - 1$, Similarly, it can be proved that $p^{\alpha} = 4(q+1)^2$ is also impossible.

We can deal with the case $p \neq 2$ similarly.

(c) If $p^{\alpha} | (q^2 + q + 1)/3$, then $P^{\alpha} < D(M)$, a contradiction.

(d) If $p^{\alpha} | 3(q-1)^2$, then $p^{\alpha} - 1 = 3D(M) - 3q - 1$, which means $3q + 1 = q^2 - q + 1$ and so $q = 4$ and $p^{\alpha} = 3^3$.

Case 2. If $q \equiv -1 \pmod 3$, then p^{α} is a factor of $4(q-1)^2$, $4(q+1)^2$, $3(q+1)^2$, and $(q^2 - q + 1)/3$. Using the same method as in Case 1, we see that $q = 5$ and $\mathrm{p}^{\alpha} = 32$.

By Tables 2 and 4 in [4] we know that if $3|q$, then $\mathrm{OC}(G) = \{q^6(q^2-1)^2, q^2+q+1, q^2-q+1\}$; if $3|q-1$, then $\mathrm{OC}(G) = \{q^6(q^2-1)^2(q^2+q+1), q^2-q+1\}$; and if $3|q+1$, then $\mathrm{OC}(G) = \{q^6(q^2-1)^2(q^2-q+1), q^2+q+1\}$. Thus if $3|q$ and $M = G_2(q)$, then it has been proved that Thompson's conjecture holds in [4]. So it is necessary to discuss the case $3 \nmid q$.

LEMMA 10. *Let G be a finite group with $Z(G) = 1$ and $M = G_2(q)$, where $3|q-1$. If $N(G) = N(M)$, then $G \cong M$.*

Proof.

Step 1. Prove that G is neither a Frobenius nor a 2-Frobenius group.

(1) Prove that G is not a Frobenius group.

If G is a Frobenius group, then, by Lemma 3, $\mathrm{OC}(G) = \{|H|, |K|\}$, where H and K are the Frobenius kernel and the Frobenius complement, respectively.

Suppose $2 \mid |H|$. Then $|K| = q^2 - q + 1$. Let p be a prime and $p | (q^2 + q + 1)/3$. Then $|S_p| < \frac{1}{3}(q^2+q+1) < q^2 - q + 1$. Since H is nilpotent, S_p is normal in G. Then $q^2 - q + 1 | |S_p|$ by Lemma 8, a contradiction.

Suppose $2 | |K|$. Then for any Sylow subgroup of H, we have $|S_p| < |K|$, contradicting $|K| \mid |S_p| - 1$ by Lemma 8.

(2) Prove that G is not a 2-Frobenius group.

If G is a 2-Frobenius group, then G has a normal series $H \trianglelefteq K \trianglelefteq G$ such that $|K/H| = q^2 - q + 1$ by Lemma 4. Since $q \equiv 1 \pmod 3$, $q \geq 4$.

If q is odd, then $(q+1)/2$ or $(q-1)/2$ is an odd integer > 2. When $(q-1)/2$ is odd, we have $((q-1)/2)^2(q^2+q+1)/3 \geq 3(q^2+q+1) > |K/H|$. By Lemma 4, $|G/K| < q^2 - q + 1$. Thus there exists a prime p

such that $p|((q-1)/2)^2(q^2+q+1)/3$, $p\,\big|\,|H|$. We shall encounter a contradiction by similar reasoning as in (1). And we can deal with the case $(q+1)/2$ odd similarly.

If q is even, then $(q-1)^2(q^2+q+1)/3 > q^2-q+1$. Taking a prime p divides $(q-1)^2(q^2+q+1)/3$, similar to (1) we also encounter a contradiction. So (2) and then Step 1 follow.

By Step 1 and Lemma 7 we know that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a simple group and $t(K/H) \geq 2$. Therefore we must check each possibility of K/H. By the classification theorem of finite simple groups and [6, 8], we may list all the possibilities of K/H as shown in Table I.

Now we must check all possibilities of K/H one by one. Since the procedure of checking is the same and trivial, we show several typical cases only.

Step 2. Prove that K/H is not an alternating group A_n.

Suppose $K/H = A_n$. Since $t(K/H) \geq 2$, we have that $n = p$, $p+1$, or $p+2$ by Table I, where p is an odd prime ≥ 5.

$$\mathrm{OC}(K/H) = \begin{cases} \{p\} & n, n-2 \textit{ not all primes} \\ \{p, p-2\} & n, n-2 \textit{ primes} \end{cases}$$

By Lemma 7 we have $q^2-q+1 = p$ or $p-2$. If $q^2-q+1 = p$, then $q^2-q-1 \,\big|\, |G|$ since $p-1 \,|\, |K/H|$. But it is easy to check that $(|G|, q^2-q-1) = 1$, a contradiction. If $q^2-q+1 = p-2$, then $p = q^2-q+3$, which implies $q^2-q+2 \,|\, (q^2+q+1)/3$ for $p \,|\, |G|$, a contradiction.

Step 3. Prove that K/H cannot be a simple group of type $A_n(q')$ or ${}^2A_n(q')$.

This step is the most difficult one; the method used here is suitable to other steps.

TABLE I

$A_n(q)$, $(n,q) = (1,q)$ or $(p-1,q)$ or (p,q) and $q-1\|p-1$	$G_2(q)$, $q > 2$
${}^2D_n(q)$, $(n,q) = (p+1,2)$ or $(2^m,q)$ or $(p,3)$, $p \geq 5$	${}^2G_2(q)$
$E_7(2)$, $E_7(3)$	${}^2B_2(q)$
A_n, $n = 5,6$ or $n = p, p+1, p+2$	${}^2F_4(2)'$
$B_n(q)$, $n = 2^m$, $m \geq 2$	$B_p(3)$, $q+2,3$
$C_n(q)$, $n = 2^m$, $m \geq 1$	$C_p(q)$, $q = 2,3$
$D_p(q)$, $p \geq 5$, $q = 3,5$	$D_{p+1}(3)$, $p \geq 3$
$E_6(q)$	$F_4(q)$, $q > 2$
${}^2A_n(q)$, $(n,q) = (p-1,q)$ or (p,q) and $q+1\|p+1$	$E_8(q)$
${}^3D_4(q)$	${}^2E_6(q)$
A sporadic simple group	$G_2(2)'$

First we prove that K/H cannot be $A_n(q')$.

If $K/H = A_n(q')$, then $(n, q') = (1, q')$ or $(p'-1, q')$ or (p', q') and $q'-1 \mid p'-1$ by Table I, where p is an odd prime.

(1) Let $n = 1$ and $q' \equiv 1 \pmod 4$. Then by [2, Table 2] the odd order components of K/H are q' and $(q'+1)/2$. Thus $q^2 - q + 1 = q'$ or $(q'+1)/2$ by Lemma 7. If $q^2 - q + 1 = q'$, then $|A_1(q')| = \frac{1}{2}q(q-1)(q^2-q+1)(q^2-q+2)$. Since $|A_1(q')| \mid |G|$, $(q^2-q+2)/2$ divides $q^6(q-1)^2(q+1)^2(q^2+q+1)(q^2-q+1)$ and so $(q^2 - q+2)/2$ divides (q^2+q+1) because $(q^2-q+2)/2$ is coprime to $q(q^2-1)$. But $q^2+q+1 = (q^2-q+2)+2q-1$. Then $(q^2 - q+2)/2$ divides $2q-1$. Because $q \equiv 1 \pmod 3$, $q \geq 4$. It is easy to check that $q = 4$; otherwise $(q^2-q+2)/2 > 2q-1$ if $q > 4$. Thus $|G| = 2^{12}\cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$ and $A_1(13) = 2^2 \cdot 3 \cdot 7 \cdot 13$. By Lemma 7, $|G/K| \mid 2$ and $5^2 \| |H|$. Since H is nilpotent, G has a normal subgroup of order 25, which leads to $13 \mid 24$ by Lemma 8, which is impossible.

(2) Let $n = 1$ and $q' \equiv -1 \pmod 4$. Then the two odd order components of K/H are q' and $(q'-1)/2$. By Lemma 7 we have $q^2 - q + 1 = q'$ or $(q'-1)/2$. We can use the same method as in (1) to deal with it if $q^2 - q + 1 = q'$. If $q^2 - q + 1 = (q'-1)/2$, then $q' \equiv 1 \pmod{D(M)}$, and so $(q, q') \neq 1$ or $q = 4$ and $q' = 3^3$ by Lemma 9. If $(q', q) \neq 1$, that is, $q = r^k$ and $q' = r^s$ for some prime r, it is easy to show this is impossible. If $q = 4$ and $q' = 27$, then $|A_1(q')| = 2^3 \cdot 3^3 \cdot 7 \cdot 13$. By similar reasoning as in (1), we have $|G/K| \mid 12$ and $5^2 \| |H|$, so that G has a normal subgroup of order 25, and then $13 \mid 24$ by Lemma 8, a contradiction.

(3) Let $n = 1$ and $4 \mid q'$. Then the odd order components of K/H are $q'+1$ and $q'-1$, so that $q^2 - q + 1 = q'+1$ or $q'-1$. Since q' is a power of 2, $q^2 - q + 1 \neq q' + 1$ and $q^2 - q + 1 = q' - 1$. Moreover $q' \equiv 1 \pmod{q^2 - q + 1}$, which means that $2 \mid q$ or $(q, q') = (4, 27)$ by Lemma 9. Obviously, it is incorrect that $(q, q') = (4, 27)$ for $4 \mid q'$. If $2 \mid q$, then $q = q' = 4$ by $q^2 - q + 1 = q' - 1$, and $|G| = 2^5 \cdot 3^3 \cdot 7$ and $|A_1(4)| = 4 \cdot 3 \cdot 5$, which is impossible by similar reasoning.

(4) Let $n = 2$ and $q' = 4$. Then K/H has odd order components 5, 7, and 9. Thus $q^2 - q + 1 = 5$, 7, or 9. It is easy to show that $q = 2$ for $q \equiv 1 \pmod 3$, a contradiction.

(5) Let $n = p - 1$ and $(n, q') \neq (2, 4)$ where p is an odd prime. Then K/H has only one odd order component $(q'^p - 1)/[(q'-1)(p, q'-1)]$ and so $(q'^p - 1)/[(q'-1)(p, q'-1)] = q^2 - q + 1$. Hence $q'^p \equiv 1 \pmod{q^2 - q + 1}$ and again $(q, q') \neq 1$ or $(q, q'^p) = (4, 27)$ by Lemma 7.

If $(q, q') \neq 1$ and $(p, q'-1) = 1$, then $q'^{p-1} + \cdots + q' + 1 = q^2 - q + 1$ and

$$q'(q'^{p-2} + \cdots + q' + 1) = q(q-1),$$

which is impossible.

If $(q, q') \neq 1$ and $(p, q'-1) = p$, then $(q'^{p-1} + \cdots + q' + 1)/p = (q'^p - 1)/(q'-1)(p, q'-1) = q^2 - q + 1$. Therefore $p - 1 \equiv 0 \pmod{q}$ or $p - 1 \equiv 0 \pmod{q'}$. Obviously, the latter one is impossible for $p \mid q' - 1$. If the first one holds and $p \geq 5$, then $q^3 > (q'^p - 1)/(q'-1)p > q'^3$. This means $q > q'$, contradicting $q \mid p - 1$ and $p \mid q' - 1$. If the first one holds and $p = 3$, then $q \mid 2$. Thus $q = 2$, a contradiction.

If $q = 4$ and $q'^p = 27$, then $q' = 3$, $p = 3$, $q^2 - q + 1 = 13$, $|G| = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$ and $|K/H| = 2^4 \cdot 3^3 \cdot 13$. By the same reasoning as in (2), we have $7 \parallel |H|$, a contradiction.

(6) Let $(n, q') = (p, q')$ and $q' - 1 \mid p + 1$. Then K/H has an odd order component $(q'^p - 1)/(q'-1)$. Similar to (5), it can be shown that this case is also impossible.

Until now we have proved that K/H cannot be $A_n(q')$. By the same method we can prove that K/H cannot be a simple group of type ${}^2A_n(q')$ because the odd order components of ${}^2A_n(q')$ are of the same form of those of $A_n(q')$.

Step 4. Prove that K/H cannot be simple groups of type $B_n(q')$ and $C_n(q')$.

Since $B_n(q')$ and $C_n(q')$ have the same order and the same odd order components, it is sufficient to discuss one of them, for example, $B_n(q')$.

If $K/H = B_n(q')$, then $(n, q') = (2^m, q')$ for $m > 1$ or (p, k) with p a prime and $k = 2$ or 3 by Table I.

(1) If $(n, q') = (2^m, q')$ with $m > 1$, then K/H has an odd order component $(q'^n + 1)/(2, q'-1)$. Therefore $(q'^n + 1)/(2, q'-1) = q^2 - q + 1$, so that $q'^{n^2} > (q(q-1))^n$. If $n \geq 8$, then $q'^{n^2} > q^6$, which is impossible by Lemma 9. Hence $n < 8$.

If $n = 4$, then $q'^{n^2} > q^4(q-1)^4$, while $q > 2$, $q'^{n^2} > q^6$, a contradiction by Lemma 9.

If $n = 2$, then $q^2 - q + 1 = (q'^2 + 1)/(2, q'-1)$. It is easy to show that $2 \nmid q'$ and $q'^4 > q^2(q-1)^2 > q^3$. Then we have $(q, q') \neq 1$ by Lemma 9. It is easy to check that this is impossible.

(2) If $(n, q') = (p, k)$ for p a prime and $k = 2$ or 3, then K/H has an odd order component $(k^p - 1)/(2, k-1)$, which is equal to $q^2 - q + 1$. By the same reasoning as in (1), we can show this is also impossible. Step 4 follows.

Step 5. Prove that K/H cannot be a simple group of type $D_n(q')$.

If $K/H = D_n(q')$, then $(n, q') = (p, k)$ or $(p' + 1, 3)$, where p is a prime ≥ 5, p' is a prime ≥ 3, and $k = 2$ or 3 or 5 by Table I.

If $(n, q') = (p, k)$, p and k as above, then K/H has an order component $(k^p - 1)/(4, k - 1)$. Thus $(k^p - 1)/(4, k - 1) = q^2 - q + 1$ and so $k^p \equiv 1 \pmod{q^2 - q + 1}$ and $k^{p(p-1)} > q^4(q-1)^4$. Then, by Lemma 7, since $q \geq 4$, $k^{p(p-1)} > q^6$ for $q > 2$, a contradiction.

If $(n, q') = (p' + 1, 3)$, then $D_n(q')$ has an odd order component $(3^{p'} - 1)/2$. For $p' \geq 5$, by the same process as in (1), it can be proved that K/H cannot be $D_n(q')$. If $p' = 3$, then $q^2 - q + 1 = 13$ and so $q = 3$ or 4. But by $q \equiv 1 \pmod 3$ we have $q = 4$, and so that $3^{12} \mid |K/H|$, contradicting $3^{12} \nmid |G|$. Step 5 follows.

Step 6. Prove that K/H cannot be a simple group of types: ${}^2B_2(q')$, ${}^2D_n(q')$, $E_6(q')$, ${}^2E_6(q')$, $F_4(q')$, ${}^2F_4(q')$, ${}^2G_2(q')$, and ${}^3D_4(q')$.

Since the processes to prove that K/H cannot be a simple group of the types listed above are the same as in those in Steps 3, 4, and 5, we omitted the proofs here.

Step 7. Prove that K/H cannot be $E_8(q')$.

By [4, Table 1], the odd order components of $E_8(q')$ are all $\leq q'^9$. If $K/H = E_8(q')$, then $q^2 - q + 1$ equals one of the odd order components of $E_8(q')$. Therefore $q'^9 > q$ and $q'^{120} > q^{10}$, which is impossible by $q'^{120} \mid |E_8(q')|$ and Lemma 7. Step 7 follows.

Step 8. Prove that K/H cannot be any one of sporadic simple groups, $E_7(2)$, $E_7(3)$, and ${}^2F_4(2)$.

By Tables 2–4 in [2] any odd order component of any simple group listed above is one of the following primes: $5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 59, 67, 71, 73, 127, 757, 1093$. Hence $q^2 - q + 1$ equals one of these primes. Then $q = 4$ by solving these equations, so that $|G| = |G_2(4)| = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$, which leads to $\operatorname{Max} \pi(K/H) = \operatorname{Max} \pi(G) = 13$. Then $K/H = Sz$ by [8]. But $11 \mid |Sz|$ and $11 \nmid |G|$, a contradiction.

Step 9. Prove that $G \cong M$.

From Steps 1–9, we know that $K/H = G_2(q')$. If $q' \equiv -1 \pmod 3$, then K/H has odd order component $q'^2 + q' + 1$, so $q'^2 + q' + 1 = q^2 - q + 1$ and $q'(q' + 1) = q(q - 1)$. Hence $q' = q - 1$ and $q' \equiv 0 \pmod 3$ for $q \equiv 1 \pmod 3$, a contradiction to $q' \equiv -1 \pmod 3$.

If $q' \equiv 0 \pmod 3$, then K/H has two odd order components $q'^2 + q' + 1$ and $q'^2 - q' + 1$. Thus $q^2 - q + 1 = q'^2 + q' + 1$ or $q'^2 - q' + 1$, *which implies* $q = q' + 1$ *or* $q = q'$. *Since* $q' \equiv 0$ (*mod* 3) *and* $q \equiv 1$ (*mod* 3), $q = q' + 1$ *and* $q'^6 = (q-1)^6$. *Therefore* $q'^6 > q^3$ *for* $q \geq 4$. *Then* $(q', q) \neq 1$ *by Lemma* 7, *a contradiction.*

If $q' \equiv 1 \pmod 3$, then $q'^2 - q' + 1 = q^2 - q + 1$, so that $q' = q$ and $G = K/H \cong M$.

This is the end of the proof of Lemma 9.

The proof of the following lemma is similar to that of Lemma 10.

LEMMA 11. *Let G be a finite group with $Z(G) = 1$ and $M = G_2(q)$, where $3 \mid q + 1$. If $N(G) = N(M)$, then $G \cong M$.*

LEMMA 12. *Let G be a finite group with $Z(G) = 1$ and $M = G_2(2)$ or $G_2(2)'$. If $N(G) = N(M)$, then $G \cong M$.*

Proof. First, since $G_2(2)' = {}^2A_2(3)$, it has been discussed in Lemma 10.

Second, by [5, Page 14] , we know that the prime graph of $G_2(2)$ has two components: $2^6 \cdot 3^3$ and 7. Then if $M = G_2(2)$, we have to repeat the processes used in the proof of Lemma 10 by Lemmas 1, 2, and 5. Similar to Step 1 in the proof of Lemma 10, we can prove that G cannot be a Frobenius group or a 2-Frobenius group. Therefore G has a normal series $H \trianglelefteq K \trianglelefteq G$, such that K/H is a simple group with $t(K/H) \geq 2$. Since $|K/H| \mid 2^6 \cdot 3^3 \cdot 7$, K/H may be one of following simple groups by [5]: $L_2(3)$ $(2^3 \cdot 3 \cdot 7)$, $L_2(8)$ $(2^3 \cdot 3^2 \cdot 7)$, and $G_2(2)'$.

If $K/H = L_2(7)$, then G/K has an order of a factor of $|\mathrm{Out}(L_2(7))|$, that is, $|G/K| \mid 2$. Hence H has a 3-Sylow subgroup of order 9 that is normal in G, a contradiction by Lemma 8. So $K/H \neq L_2(7)$.

If $K/H = L_2(8)$, then $|G/K| \mid 3$ by Lemma 5. Therefore the 2-Sylow subgroup N of H is of order 8. by Lemma 8, N has exactly two conjugacy classes, one containing seven elements and one element. That is to say, for any $x \in N$ and $x \neq 1$, we have $|\mathrm{Cl}_G(x)| = 7$, contradicting $7 \notin N(G_2(2))$ by [5, p. 13]. Hence $K/G \neq L_2(8)$.

If $K/H = G_2(2)'$, then $|H| = 1$ or 2. By Lemma 8, $|H| = 1$ and $|G/K| = 2$, which leads to $G \cong M$.

Proof of the Main Theorem. The theorem follows from Lemmas 10–12.

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